Weak Solutions and Their Numerical Analysis of Nonlinear Parabolic Equations of Forth Order based on FEM (Dynamics of Functional Equations and Related Topics)

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Weak Solutions and Their Numerical Analysis of Nonlinear Parabolic Equations of Fourth Order based on FEM

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1 Introduction

In this paper we investigate the weak solutions and their numerical analysis of nonlinear parabolic equation of fourth order. In recent years, there are many mathematical literature concerning with nonnegative or positive solutions to fourth-order parabolic equations (cf. [1], [3]). However, the study of numerical analysis of nonlinear fourth order parabolic equations is few. In [5], we studied abstract nonlinear parabolic equations having uniform Lipschitz continuous nonlinearities, but the fourth order equations are not treated in [5]. The purpose of this paper is to study the weak and numerical solutions of fourth order parabolic equations which include nonlinear gradient and Laplacian terms.

Let Ω be an open bounded domain of $\mathbf{R}^m$ and $\partial\Omega=\Gamma$ be the piecewise smooth boundary of Ω. Let $T>0$, $Q=(0,T)\times\Omega$ and $\Sigma=(0,T)\times\Gamma$. We consider the following nonlinear parabolic equation of fourth order

$$\frac{\partial y}{\partial t}+\Delta(a(t,x)\Delta y)=f(t,x,y,\nabla y,\Delta y) \quad \text{in } Q, \tag{1.1}$$

where $a\in C([0,T];L^\infty(\Omega))$ satisfies $a(t,x)\geq A>0$ for all $(t,x)\in Q$ and $f\in L^\infty([0,T]\times\Omega\times\mathbf{R}\times\mathbf{R}^n\times\mathbf{R})$ is a nonlinear forcing function. The initial condition is given by $y(0,x)=y_0(x)$ in Ω. The attached boundary condition is given by the one of the following four types of conditions (cf. Dautray and Lions [2]).

Case 1 (Dirichlet boundary condition) $\quad y(t,x)=\dfrac{\partial}{\partial n}y(t,x)=0 \quad \text{on } \Sigma;$ (1.2.1)

Case 2 (Neumann boundary condition) $\quad \Delta y(t,x)=\dfrac{\partial}{\partial n}\Delta y(t,x)=0 \quad \text{on } \Sigma;$ (1.2.2)

Case 3 (Mixed boundary condition, A) $\quad y(t,x)=\Delta y(t,x)=0 \quad \text{on } \Sigma;$ (1.2.3)

Case 4 (Mixed boundary condition, B) $\quad \dfrac{\partial y}{\partial \eta}(t,x)=\dfrac{\partial}{\partial \eta}(a(t,x)\Delta y(t,x))=0 \quad \text{on } \Sigma.$ (1.2.4)

We explain the content of this paper. In section 2, we prove the existence and uniqueness theorem of weak solutions for the problem (1.1) with one of (1.2.1)-(1.2.4). At the same time we give the estimate of weak solutions with respect to initial values and forcing terms. After this, we study the numerical analysis of the problem based on the finite element method in section 3. As numerical simulations we consider the special case where $a(t,x)\equiv 1$ and $f(t,x,y,\nabla y,\Delta y)=\alpha\sin y+\beta\sin\nabla y+\gamma\sin\Delta y$.

2 Existence and Uniqueness of Weak Solutions

In this section, we study the existence and uniqueness of weak solutions for the initial-boundary value problem (1.1) with one of (1.2.1)-(1.2.4). In order to solve the problem in the framework of variational method due to Dautray and Lions [2], we introduce two Hilbert space $H = L^2(\Omega)$ and the maximum domain $H(\Delta;\Omega) = \{\phi \in L^2(\Omega) \mid \Delta\phi \in L^2(\Omega)\}$. $H(\Delta;\Omega)$ is a Hilbert space with the inner product $(\phi,\psi)_{H(\Delta;\Omega)} = (\phi,\psi) + (\Delta\phi,\Delta\psi)$, where $(\ ,\ )$ is the inner product of $H = L^2(\Omega)$. We now take the pivot Hilbert space (specified later) V such as $H_0^2(\Omega) \subset V \subset H(\Delta;\Omega)$. Thus V is a closed subspace of $H(\Delta;\Omega)$ equipped with the norm

$$\|\phi\| = (|\phi|^2 + |\Delta\phi|^2)^{\frac{1}{2}}, \quad |\phi| = \left(\int_\Omega |\phi(x)|^2 dx\right)^{\frac{1}{2}}. \tag{2.1}$$

We note that the norm $\|\cdot\|$ is equivalent to the norm of $H^2(\Omega)$, i.e. there exists a $c_1 > 0$ such that

$$\|\phi\|_{H^2(\Omega)} \le c_1\|\phi\|, \quad \forall\phi \in H^2(\Omega). \tag{2.2}$$

For such a V we define the space

$$W(0,T) = \{g \mid g \in L^2(0,T;V), g' \in L^2(0,T;V')\}. \tag{2.3}$$

We introduce the bilinear form

$$a(t;\phi,\psi) = \int_\Omega a(t,x)\Delta\phi(x)\Delta\psi(x)dx, \quad \forall\phi,\psi \in V \subset H(\Delta;\Omega). \tag{2.4}$$

associated with the fourth order differential operator $\Delta(a(t,x)\Delta)$. It is clear that $a(t;\phi,\phi) \ge A|\Delta\phi|^2$, $\forall t \in [0,T]$. Further we suppose that for any $\phi \in H(\Delta;\Omega)$ the function $f(t;\phi) = f(t,x,\phi,\nabla\phi,\Delta\phi)$ defines a function in $H = L^2(\Omega)$ for each $t \in [0,T]$. Here we take V as follows for the case 1-4.

$$\begin{array}{llll} \text{Case 1:} & V = H_0^2(\Omega), & \text{Case 2:} & V = H(\Delta;\Omega), \\ \text{Case 3:} & V = \{\phi \in H(\Delta;\Omega) \mid \phi|_\Gamma = 0\}, & \text{Case 4:} & V = \{\phi \in H(\Delta;\Omega) \mid \dfrac{\partial\phi}{\partial n}|_\Gamma = 0\}. \end{array} \tag{2.5}$$

Now we give the definition of weak solutions for the problem (1.1) with one of (1.2.1)-(1.2.4), and shortly we shall call the problem (P).

Definition 1 A function y is said to be a weak solution of the problem (P) if $y \in W(0,T)$ and y satisfies

$$\begin{cases} \langle y'(\cdot),\phi\rangle_{V',V} + a(\cdot;y(\cdot),\phi) = (f(\cdot;y(\cdot)),\phi) & \text{for all } \phi \in V \text{ in the sense of } \mathcal{D}'(0,T), \\ y(0) = y_0, \end{cases} \tag{2.6}$$

Where V is given by the one indicated in (2.6), the symbol $\langle\cdot,\cdot\rangle_{V',V}$ denotes a dual pairing between V and V', and $\mathcal{D}'(0,T)$ denotes the space of distributions on $(0,T)$.

Assume that $f : [0,T] \times \Omega \times \mathbf{R} \times \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ satisfy

(i) $f(\cdot,x,y,\xi,\eta)$ is measurable on $[0,T]$ for each $x \in \Omega$, $y \in \mathbf{R}$, $\xi \in \mathbf{R}^n$ and $\eta \in \mathbf{R}$;

(ii) $f(\cdot,x,y,\xi,\eta)$ is measurable on Ω for each $t \in [0,T]$, $y \in \mathbf{R}$, $\xi \in \mathbf{R}^n$ and $\eta \in \mathbf{R}$;

(iii) there is a $c \in L^\infty(Q)$ such that for $\forall(t,x) \in Q$, $\forall y, y', \xi, \xi' \in \mathbf{R}$, $\forall \eta, \eta' \in \mathbf{R}^n$

$$|f(t,x,y,\xi,\eta) - f(t,x,y,\xi',\eta')| \le c(t,x)(|y-y'| + |\xi - \xi'| + |\eta - \eta'|);$$

(iv) there is a $\gamma \in L^2(Q)$ such that $|f(t,x,0,0,0)| \le \gamma(t,x)$, $\forall(t,x) \in Q$.

Theorem 1 Assume that f satisfies **(i)-(iv)**. Then for $y_0 \in L^2(\Omega)$, there exists a unique weak solution $y \in W(0,T)$ of (P) such that $y \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega))$. Further the estimate

$$\|y\|^2_{L^\infty(0,T;H)} + \|y\|^2_{L^2(0,T;H^2(\Omega))} \le C(|y_0|^2 + \|\gamma\|^2_{L^2(Q)})exp(C\|c\|^2_{L^\infty(Q)}) \tag{2.7}$$

holds for some $C > 0$ indenpent of y_0.

Proof. Define the function $\bar{f} : [0,T] \times V \to H$ by $\bar{f}(t,\phi)(x) = f(t,\phi(x),\nabla\phi(x),\Delta\phi(x))$, *a.e.* $x \in \Omega$. Then by **(iii)** and (2.2), we have

$$\begin{aligned}
&|\bar{f}(t,y_1) - \bar{f}(t,y_2)|^2 = |f(t;y_1) - f(t;y_2)|^2_H \\
= \;& \int_\Omega |f(t,x,y_1,\nabla y_1,\Delta y_1) - f(t,x,y_2,\nabla y_2,\Delta y_2)|^2 dx \\
\le \;& 2\|c\|^2_{L^\infty(Q)} \int_\Omega (|y_1 - y_2|^2 + |\nabla y_1 - \nabla y_2|^2 + |\Delta y_1 - \Delta y_2|^2) dx \\
\le \;& 2\|c\|^2_{L^\infty(Q)} \|y_1 - y_2\|^2_{H^2(\Omega)} \le 2c_1^2 \|c\|^2_{L^\infty(Q)} \|y_1 - y_2\|^2.
\end{aligned} \tag{2.8}$$

This proves that the nonlinear term in (2.6) satisfies the uniform Lipschitz continuity. Hence by Wang and Nakagiri [5], there exists a unique weak solution $y \in W(0,T)$ of the problem (P) under the assumptions **(i)-(iv)**.

Next we shall prove the estimate (2.7). Taking $\phi = y$ in the weak form (2.6) and integrating them on $[0,t]$, by **(iii)**, **(iv)** and (2.8) we have

$$\begin{aligned}
\frac{1}{2}|y(t)|^2 + A\int_0^t |\Delta y| dt \;&\le\; \frac{1}{2}|y(0)|^2 + \int_0^t |f(s;y) - f(s;0)|\,|y(s)|ds + \int_0^t |f(s;0)||y(s)|ds \\
&\le\; \frac{1}{2}|y(0)|^2 + \sqrt{2}c_1\|c\|_{L^\infty(Q)} \int_0^t \|y(s)\|\,|y(s)|ds + \int_0^t |\gamma(s,\cdot)||y(s)|ds.
\end{aligned}$$

Hence, for any $\varepsilon > 0$, we have

$$\frac{1}{2}|y(t)|^2 + (A-\varepsilon)\int_0^t \|y(s)\|^2 ds \le \frac{1}{2}|y_0|^2 + \frac{1}{2}\|\gamma\|^2_{L^2(Q)} + (\frac{2}{\varepsilon}c_1^2\|c\|^2_{L^\infty(Q)} + 2)\int_0^t |y(s)|^2 ds. \tag{2.9}$$

By setting $\varepsilon = \dfrac{A}{2}$ and applying the Bellmann-Gronwall inequality to (2.9), we have

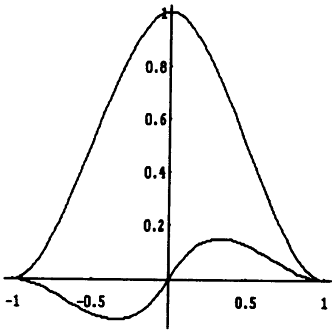
$$|y(t)|^2 + \|y\|^2_{L^2(0,T;V)} \le C(|y_0|^2 + \|\gamma\|^2_{L^2(Q)})exp(C\|c\|^2_{L^\infty(Q)}), \quad \forall t \in [0,T] \tag{2.10}$$

for some $C > 0$. Hence (2.7) follows. This completes the proof.

3 Numerical Analysis based on FEM

In this section, we study the numerical analysis of one dimensional nonlinear fourth order parabolic equations (1.1) based on the finite element method. We construct a rather complete and effective algorithm for approximate solutions by using the cubic base functions for each type of boundary conditions. The difference depends on the choice of the basis of V. Using *Mathematica*, we give several figures of weak solutions for different types of initial data, forcing functions and physics parameters.

Let $0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = l$ be a partition of the interval $[0, l]$ into subinterval $I_e = [x_{e-1}, x_e]$ of length $h_e = x_e - x_{e-1}, e = 1, 2, \cdots, N+1$. Let V_h be the set of functions such that ϕ is cubic on each I_e and is continuous on $[0, l]$. Then it is clear that $V_h \subset H_0^2(0, l)$. Let us introduce the base functions ψ_i^e defined by cubic interpolation functions, which can be expressed as



$$\begin{cases} \psi_1^e(x) = 1 - \dfrac{3}{h_e^2}(x - x_e)^2 + \dfrac{2}{h_e^3}(x - x_e)^3, \\ \psi_2^e(x) = (x - x_e) - \dfrac{2}{h_e}(x - x_e)^2 + \dfrac{1}{h_e^2}(x - x_e)^3, \\ \psi_3^e(x) = \dfrac{3}{h_e^2}(x - x_e)^2 - \dfrac{2}{h_e^3}(x - x_e)^3, \\ \phi_4^e(x) = -\dfrac{1}{h_e}(x - x_e)^2 + \dfrac{1}{h_e^2}(x - x_e)^3. \end{cases}$$

The Hermite cubic interpolation functions satisfy the following interpolation properties

$$\begin{array}{lll} \psi_1^e(x_e) = 1, & \psi_i^e(x_e) = 0 & (i \neq 1), \\ \psi_3^e(x_{e+1}) = 1, & \psi_i^e(x_{e+1}) = 0 & (i \neq 3), \\ \left(\dfrac{-d\psi_2^e}{dx}\right)\Big|_{x_e} = 1, & \left(\dfrac{d\psi_i^e}{dx}\right)\Big|_{x_e} = 0 & (i \neq 2), \\ \left(\dfrac{-d\psi_4^e}{dx}\right)\Big|_{x_{e+1}} = 1, & \left(\dfrac{d\psi_i^e}{dx}\right)\Big|_{x_{e+1}} = 0 & (i \neq 4). \end{array} \tag{3.1}$$

We give the analysis only for the Case 1: $V = H_0^2(0, l)$. We omit others cases here. Case 1 corresponds to the following one dimensional initial boundary value problem

$$\begin{cases} \dfrac{\partial y}{\partial t} + \Delta(a(t, x)\Delta y) = f(t, x, y, \nabla y, \Delta y), \quad \text{in } (0, l) \times (0, T), \\ y(t, 0) = \dfrac{\partial y}{\partial n}(t, 0) = y(t, l) = \dfrac{\partial y}{\partial n}(t, l) = 0, \quad \text{on } (0, T), \\ y(0, x) = y_0(x), \quad a.e. \text{ on } (0, l). \end{cases} \tag{3.2}$$

The e-th element of approximate solution for (1.1) is defined by $y_h^e(t, x) = \sum_{i=1}^{4} \xi_i^e(t)\psi_i^e(x)$, $e = 1, 2, \cdots, N$. Then the total approximate solution can be represented as

$$y_h(t, x) = \sum_{e=1}^{N} y_h^e(t, x) = \sum_{e=1}^{N}\sum_{i=1}^{4} \xi_i^e(t)\psi_i^e(x) \in V_h \subset V, \quad \forall t \in [0, T],$$

where y_h^e, $e = 1, 2, \cdots, N$ satisfies

$$\begin{cases} ((y_h^e)', \psi_j^e) + (a(t, \cdot)\Delta y_h^e, \Delta\psi_j^e) = (f(t, \cdot, y_h^e, \nabla y_h^e, \Delta y_h^e), \psi_j^e), \\ (y_h^e(0), \psi_j^e) = (y_0, \psi_j^e). \end{cases} \tag{3.3}$$

We can rewrite (3.3) as follows:

$$\begin{cases} \sum_{i=1}^{4} \xi_i^{e\prime}(t)(\psi_i^e, \psi_j^e) + \sum_{i=1}^{4} \xi_i^e(t)(a(t,\cdot)\Delta\psi_i^e, \Delta\psi_j^e) - (f(t,\cdot,y_h^e, \nabla y_h^e, \Delta y_h^e), \psi_j^e) = 0, \\ \sum_{i=1}^{4} \xi_i^e(0)(\psi_i^e, \psi_j^e) = (y_0, \psi_j^e), \qquad e = 1, 2, \cdots, N. \end{cases} \tag{3.4}$$

By the interpolation properities (3.1), we set $\psi_1^1 = 0, \psi_3^N = 0$ and $\nabla\psi_2^1 = 0, \nabla\psi_4^N = 0$. For simplicity we denote $\nabla\psi = \dot{\psi}$ and $\Delta\psi = \ddot{\psi}$. Then the first equation of (3.4) can be written as

$$\sum_{i=1}^{4} \xi_i^{e\prime}\psi_{ij}^e + \sum_{i=1}^{4} \xi_i^e \phi_{ij}^e - f_j^e = 0, \tag{3.5}$$

where

$$\psi_{ij}^e = (\psi_i^e, \psi_j^e), \quad \phi_{ij}^e = (a(t,\cdot)\ddot{\psi}_i^e, \ddot{\psi}_j^e), \quad f_j^e = (f^e(t,\cdot,y_h^e, \dot{y_h}^e, \ddot{y_h}^e), \psi_j^e).$$

Now we set

$$\begin{aligned} &\Psi^e = (\psi_i^e, \psi_j^e)_{i=1,2,3,4}^{j=1,2,3,4} \in M_{4\times4}(\mathbf{R}), \\ &\Phi^e(t) = (a(t,\cdot)\ddot{\psi}_i^e, \ddot{\psi}_j^e)_{i=1,2,3,4}^{j=1,2,3,4} \in M_{4\times4}(\mathbf{R}), \\ &\Xi^e(t) = [\xi_1^e(t), \xi_2^e(t), \xi_3^e(t), \xi_4^e(t)]^T \in M_{4\times1}(\mathbf{R}), \\ &Y_0^e = [(y_0, \psi_1^e), (y_0, \psi_2^e), (y_0, \psi_3^e), (y_0, \psi_4^e)]^T \in M_{4\times1}(\mathbf{R}). \end{aligned}$$

$$F^e(t, \Xi^e(t)) = \begin{bmatrix} (f(t,\cdot,\sum_{i=1}^{4}\xi_i^e(t)\psi_i^e, \sum_{i=1}^{4}\xi_i^e(t)\dot{\psi}_i^e, \sum_{i=1}^{4}\xi_i^e(t)\ddot{\psi}_i^e), \psi_1^e) \\ (f(t,\cdot,\sum_{i=1}^{4}\xi_i^e(t)\psi_i^e, \sum_{i=1}^{4}\xi_i^e(t)\dot{\psi}_i^e, \sum_{i=1}^{4}\xi_i^e(t)\ddot{\psi}_i^e), \psi_2^e) \\ (f(t,\cdot,\sum_{i=1}^{4}\xi_i^e(t)\psi_i^e, \sum_{i=1}^{4}\xi_i^e(t)\dot{\psi}_i^e, \sum_{i=1}^{4}\xi_i^e(t)\ddot{\psi}_i^e), \psi_3^e) \\ (f(t,\cdot,\sum_{i=1}^{4}\xi_i^e(t)\psi_i^e, \sum_{i=1}^{4}\xi_i^e(t)\dot{\psi}_i^e, \sum_{i=1}^{4}\xi_i^e(t)\ddot{\psi}_i^e), \psi_4^e) \end{bmatrix} \in M_{4\times1}(\mathbf{R}).$$

Then (3.5) can be rewriten as

$$\Psi^e\Xi^{e\prime}(t) + \Phi^e(t)\Xi^e(t) - F^e(t, \Xi^e(t)) = 0. \tag{3.6}$$

We get the whole assembled system equation

$$\Psi\Xi'(t) + \Phi\Xi(t) - \bar{F}(t, \Xi(t) = 0. \tag{3.7}$$

Here in (3.7), by taking into account of boundary condition in (3.2), we set

$$\Xi = [0, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6, \cdots, \xi_{2N-3}, \xi_{2N-2}, \xi_{2N-1}, \xi_{2N}, 0, \xi_{2N+2}]^T,$$

where

$$\begin{aligned} &\xi_1 = \xi_1^1 = 0, \ \xi_2 = \xi_2^1, \ \xi_3 = \xi_3^1 = \xi_1^2, \ \xi_4 = \xi_4^1 = \xi_2^2 \\ &\xi_{2i-3} = \xi_3^{i-2} = \xi_1^{i-1}, \ \xi_{2i-2} = \xi_4^{i-2} = \xi_2^{i-1}, \ \xi_{2i-1} = \xi_3^{i-1} = \xi_1^i, \ \xi_{2i} = \xi_4^{i-1} = \xi_2^i, \ i = 3, ..., N \\ &\xi_{2N-1} = \xi_3^{N-1} = \xi_1^N, \ \xi_{2N} = \xi_4^{N-1} = \xi_2^N, \ \xi_{2N+1} = \xi_3^N = 0, \ \xi_{2N+2} = \xi_4^N. \end{aligned}$$

In what follows we set $h_e = h$ and $a(t,x) \equiv 1$. The components of $\bar{F}$ can be approximated by applying the 6-th order Gauss-Legendre quadrature at six points $p_1^e, p_2^e, \cdots, p_6^e$ with weights $w_1^e, w_2^e, \cdots, w_6^e$ on each interval I_e. Then Ψ, Φ and $\bar{F}$ can be calculated as follows:

$$\Psi = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{h^3}{105} & \frac{13\,h^2}{420} & \frac{-h^3}{140} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{13\,h^2}{420} & \frac{26\,h}{35} & 0 & \frac{9\,h}{70} & \frac{-13\,h^2}{420} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-h^3}{140} & 0 & \frac{2h^3}{105} & \frac{13\,h^2}{420} & \frac{-h^3}{140} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{9\,h}{70} & \frac{13\,h^2}{420} & \frac{26\,h}{35} & 0 & \frac{9\,h}{70} & \frac{-13\,h^2}{420} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{-13\,h^2}{420} & \frac{-h^3}{140} & 0 & \frac{2h^3}{105} & \frac{13\,h^2}{420} & \frac{-h^3}{140} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \frac{9\,h}{70} & \frac{13\,h^2}{420} & \frac{26\,h}{35} & 0 & 0 & \frac{-13\,h}{420} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \frac{-13\,h^2}{420} & \frac{-h^3}{140} & 0 & \frac{2h^3}{105} & 0 & \frac{-h^3}{140} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{-13\,h^2}{420} & \frac{-h^3}{140} & 0 & \frac{h^3}{105}
\end{bmatrix}$$

$$\Phi = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{24}{h^3} & 0 & -\frac{12}{h^3} & -\frac{6}{h^2} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{8}{h} & \frac{6}{h^2} & \frac{2}{h} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{12}{h^3} & \frac{6}{h^2} & \frac{24}{h^3} & 0 & -\frac{12}{h^3} & -\frac{6}{h^2} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{6}{h^2} & \frac{2}{h} & 0 & \frac{8}{h} & \frac{12}{h^3} & \frac{6}{h^2} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -\frac{12}{h^3} & \frac{6}{h^2} & \frac{24}{h^3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -\frac{6}{h^2} & \frac{2}{h} & 0 & \frac{8}{h} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$F(t,\Xi(t)) = \left[\begin{array}{c}
0 \\
\sum_{j=1}^{6} w_j^1 f(t, p_j^1, \sum_{i=1}^{4} \xi_i^1 \psi_i^1(p_j^1)), \sum_{i=1}^{4} \xi_i^e \dot{\psi}_i^1(p_j^1), \sum_{i=1}^{4} \xi_i^e \ddot{\psi}_i^1(p_j^1))\psi_2^1(p_j^1) \\
\sum_{j=1}^{6} w_j^1 f(t, p_j^1, \sum_{i=1}^{4} \xi_i^1 \psi_i^1(p_j^1), \sum_{i=1}^{4} \xi_i^1 \dot{\psi}_i^1(p_j^1), \sum_{i=1}^{4} \xi_i^1 \ddot{\psi}_i^1(p_j^1))\psi_3^1(p_j^1) + \sum_{j=1}^{6} w_j^2 f(t, p_j^2, \sum_{i=1}^{4} \xi_i^2 \psi_i^2(p_j^2), \sum_{i=1}^{4} \xi_i^2 \dot{\psi}_i^2(p_j^2), \sum_{i=1}^{4} \xi_i^2 \ddot{\psi}_i^2(p_j^2))\psi_1^2(p_j^2) \\
\sum_{j=1}^{6} w_j^1 f(t, p_j^1, \sum_{i=1}^{4} \xi_i^1 \psi_i^1(p_j^1), \sum_{i=1}^{4} \xi_i^1 \dot{\psi}_i^1(p_j^1), \sum_{i=1}^{4} \xi_i^1 \ddot{\psi}_i^1(p_j^1))\psi_4^1(p_j^1) + \sum_{j=1}^{6} w_j^2 f(t, p_j^2, \sum_{i=1}^{4} \xi_i^2 \psi_i^2(p_j^2)), \sum_{i=1}^{4} \xi_i^2 \dot{\psi}_i^2(p_j^2), \sum_{i=1}^{4} \xi_i^2 \dot{\psi}_i^2(p_j^2))\psi_2^2(p_j^2) \\
\sum_{j=1}^{6} w_j^2 f(t, p_j^2, \sum_{i=1}^{4} \xi_i^2 \psi_i^2(p_j^2), \sum_{i=1}^{4} \xi_i^2 \dot{\psi}_i^2(p_j^2), \sum_{i=1}^{4} \xi_i^2 \ddot{\psi}_i^2(p_j^2))\psi_3^2(p_j^2) + \sum_{j=1}^{6} w_j^3 f(t, p_j^3, \sum_{i=1}^{4} \xi_i^3 \psi_i^3(p_j^3), \sum_{i=1}^{4} \xi_i^3 \dot{\psi}_i^3(p_j^3), \sum_{i=1}^{4} \xi_i^3 \ddot{\psi}_i^3(p_j^3))\psi_1^3(p_j^3) \\
\sum_{j=1}^{6} w_j^2 f(t, p_j^2, \sum_{i=1}^{4} \xi_i^2 \psi_i^2(p_j^2), \sum_{i=1}^{4} \xi_i^2 \dot{\psi}_i^2(p_j^2), \sum_{i=1}^{4} \xi_i^2 \ddot{\psi}_i^2(p_j^2))\psi_4^2(p_j^2) + \sum_{j=1}^{6} w_j^3 f(t, p_j^3, \sum_{i=1}^{4} \xi_i^3 \psi_i^3(p_j^3), \sum_{i=1}^{4} \xi_i^3 \dot{\psi}_i^3(p_j^3), \sum_{i=1}^{4} \xi_i^3 \ddot{\psi}_i^3(p_j^3))\psi_2^3(p_j^3) \\
\vdots \\
\sum_{j=1}^{6} w_j^{N-1} f(t, p_j^{N-1}, \sum_{i=1}^{4} \xi_i^{N-1} \psi_i^{N-1}(p_j^{N-1}), \sum_{i=1}^{4} \xi_i^{N-1} \dot{\psi}_i^{N-1}(p_j^{N-1}), \sum_{i=1}^{4} \xi_i^{N-1} \ddot{\psi}_i^{N-1}(p_j^{N-1}))\psi_3^{N-1}(p_j^{N-1}) \\
+ \sum_{j=1}^{6} w_j^N f(t, p_j^e, \sum_{i=1}^{4} \xi_i^N \psi_i^N(p_j^N), \sum_{i=1}^{4} \xi_i^N \dot{\psi}_i^N(p_j^N), \sum_{i=1}^{4} \xi_i^N \ddot{\psi}_i^N(p_j^N))\psi_1^N(p_j^N) \\
\sum_{j=1}^{6} w_j^{N-1} f(t, p_j^N, \sum_{i=1}^{4} \xi_i^{N-1} \psi_i^{N-1}(p_j^{N-1}), \sum_{i=1}^{4} \xi_i^{N-1} \dot{\psi}_i^{N-1}(p_j^{N-1}), \sum_{i=1}^{4} \xi_i^{N-1} \ddot{\psi}_i^{N-1}(p_j^{N-1}))\psi_4^{N-1}(p_j^{N-1}) \\
+ \sum_{j=1}^{6} w_j^N f(t, p_j^N, \sum_{i=1}^{4} \xi_i^N \psi_i^N(p_j^N), \sum_{i=1}^{4} \xi_i^N \dot{\psi}_i^N(p_j^N), \sum_{i=1}^{4} \xi_i^N \ddot{\psi}_i^N(p_j^N))\psi_2^N(p_j^N) \\
0 \\
\sum_{j=1}^{6} w_j^N f(t, p_j^N, \sum_{i=1}^{4} \xi_i^N \psi_i^N(p_j^N), \sum_{i=1}^{4} \xi_i^N \dot{\psi}_i^N(p_j^N), \sum_{i=1}^{4} \xi_i^N \ddot{\psi}_i^N(p_j^N))\psi_4^N(p_j^N)
\end{array}\right]$$

We can solve the degenerate first order differential equation (3.7) by taking the re
f Ψ, Φ and $\bar{F}$ and using the Runge-Kutta method of fourth order.

Simulation results

. Case of $f(t,x,y,\nabla y,\Delta y) = \alpha \sin y$. Let $l = 1$ and $y_0(x) = \sin(\pi x)$.

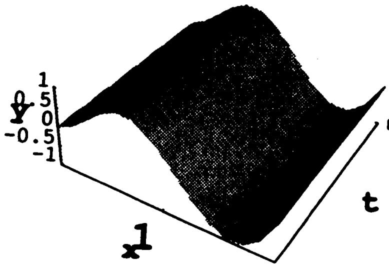


*Fig.*1 $\alpha = 0.0001$

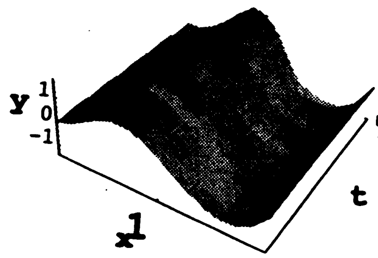


*Fig.*2 $\alpha = 0.5$

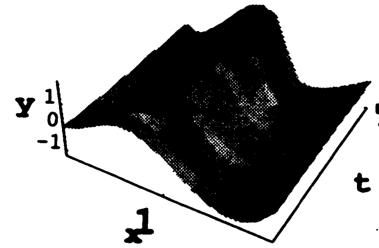


*Fig.*3 $\alpha = 1.0$

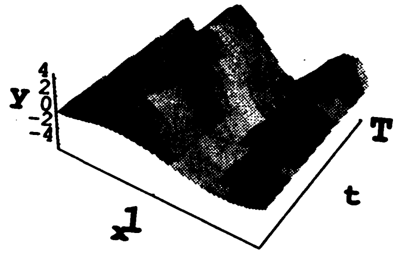


*Fig.*4 $\alpha = 5.0$

2. Case of $f(t,x,y,\nabla y,\Delta y) = \beta \sin(\Delta y)$. Let $l = 1$ and $y_0(x) = \cos(\pi x)$.

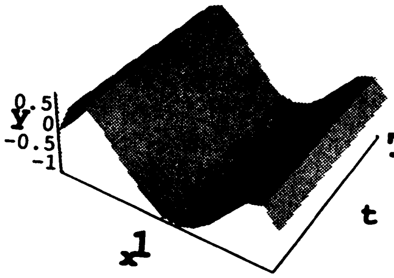


*Fig.*1 $\beta = 0.0001$

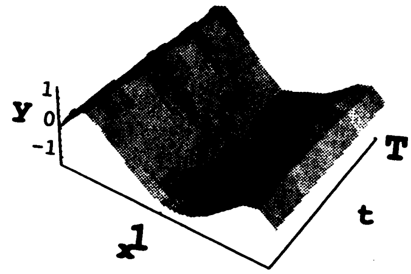


*Fig.*2 $\beta = 0.1$

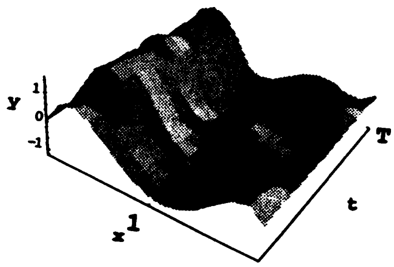


*Fig.*3 $\beta = 0.5$

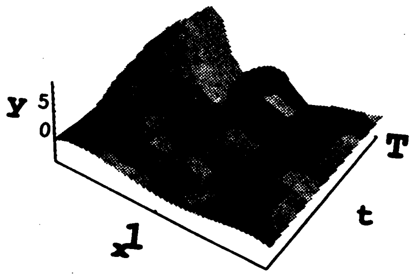


*Fig.*4 $\beta = 1.0$

3. Case of $f(t,x,y,\nabla y,\Delta y)=\gamma\sin(\Delta y)$. Let $l=1$ and $y_0(x)=\sin^2(\pi x)$.

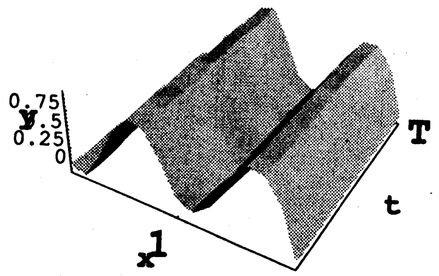


Fig.1 $\gamma=0.0001$

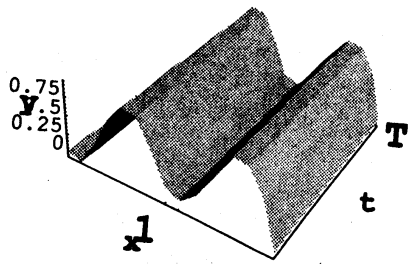


Fig.2 $\gamma=0.1$

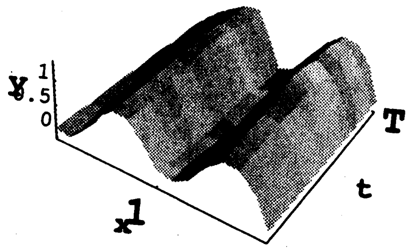


Fig.3 $\gamma=0.5$

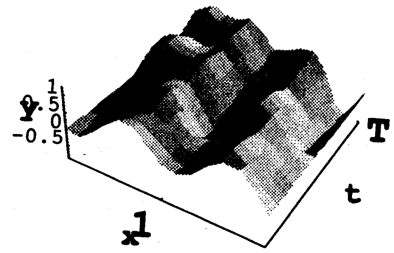


Fig.4 $\gamma=1.0$

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